B.Sc Part - II Test Examination 2013
BANGABASI COLLEGE
MATHEMATICS (HONOURS)
PAPER - IV

Time - 4hr F.M. — 100

MODULE — VII
(Application of Calculus, Real valued functions of Several variables)

1. Answer any four questions 4x5

(a) Prove that the pedal equation of the curve $c^2(x^2+y^2) = x^2y^2$ with respect to the origin is $\frac{1}{p^2} + \frac{3}{r^2} = \frac{1}{c^2}$

(b) If ρ_1, ρ_2 be the radii of curvature at the extremities of any chord of the cardioide $r = a(1+\cos\theta)$ which passes through the pole, then show that $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$

(c) Prove that the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is
$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}}.$$

(d) Given that $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$ is the envelope of the family of straight lines $\frac{x}{a} + \frac{y}{b} = 1$, find the necessary relation between a and b

(e) Find the asymptotes of the curve
$$(x+y)(x-2y)(x-y)^2 + 3xy(x-y) + x^2+y^2 = 0$$

(f) Find the range of values of x for which the curve
$$y = x^4 - 6x^3 + 12x^2 + 5x + 7$$
is concave upwards or downwards. Also find the points of ~~inflexion.~~ inflexion.

(g) Find the area included between the curve $x^2y^2 = a^2(y^2-x^2)$ and its asymptotes.

2. Answer any six questions 6x5

(a) When is the point (a,b) said to be a limiting point of a subset A of $R \times R$? If $B = \{(a,0); a \in R\}$, show that B is a closed subset but not an open subset of $R \times R$.

(b) If $f(x,y)$ is continuous at a point (a,b) of the domain, prove that the functions g and h defined by $g(x) = f(x,b)$ and $h(y) = f(a,y)$ are continuous at $x=a$ and $y=b$ respectively

(c) If $f(x,y)$ is continuous at a point (a,b) of its domain and if $f(a,b) \neq 0$, then show that $f(x,y)$ has the same sign as $f(a,b)$ in some neighbourhood of (a,b)

(d) Show that the function f, where $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } x^2+y^2\neq 0\\ 0 & \text{if } x=y=0\end{cases}$

is continuous, possesses partial derivatives but is not differentiable at the origin.

(e) Let $f(x,y)=\begin{cases} x^2\sin\frac{1}{x}+y^2\sin\frac{1}{y} & \text{when } x\neq 0,\ y\neq 0\\ x^2\sin\frac{1}{x} & \text{when } x\neq 0,\ y=0\\ y^2\sin\frac{1}{y} & \text{when } x=0,\ y\neq 0\\ 0 & \text{when } x=0,\ y=0\end{cases}$

Show that $f_x(x,y)$ and $f_y(x,y)$ are discontinuous at $(0,0)$ but that $f(x,y)$ is differentiable at $(0,0)$.

(f) If u be a homogeneous function of degree n (possessing continuous second order partial derivative) then prove the following $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$. Is the result valid if we assume the mere existence of second order partial derivatives? Justify your answer.

(g) State Euler's Theorem on homogeneous function of two variables. If $u=\tan^{-1}\frac{x^3+y^3}{x-y}$, show that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(1-4\sin^2 u)\sin 2u$. (The relevent partial derivatives are assumed to be continuous).

(h) If z is a function of two variables x, y and $x=c\cosh u\cos v$, $y=c\sinh u\sin v$ (c is a real no.) show that

$$\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}=\frac{1}{2}c^2(\cosh 2u-\cos 2v)\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)$$

(assume that second order partial derivatives of z are continuous)

(i) Show that the three functions u, v, w given by $u=3x+2y-z$, $v=x-2y+z$, $w=x(x+2y-z)$ are connected by a functional equation and find that equation.

(j) State the implicit function theorem for a function of two variables. Hence prove that $2xy-\log_e(xy)=2e-1$ determines y uniquely as a function of x near the point $(1,e)$ and find $\frac{dy}{dx}$ at $(1,e)$.

MODULE - VIII

(Analytical geometry of Three dimensions, Analytical statics, Analytical Dynamics of a particle)

3. Answer any three questions 3×5

(a) A variable sphere passes through the points $(0,0,\pm c)$ and cuts the straight lines $y=x\tan\alpha$, $z=c$ and $y=-x\tan\alpha$, $z=-c$ at the points P and P′ respectively. If $PP' = 2a$, a constant, show that the centre of the sphere lies on the circle $z=0,\ x^2+y^2 = (a^2-c^2)\operatorname{Cosec}^2 2\alpha$

(b) Show that only one tangent plane can be drawn to the sphere $x^2+y^2+z^2-2x+6y+2z+8=0$ through the line $3x-4y-8=0,\ y-3z+2=0$. Find the equation of the plane.

(c) Prove that the plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in two perpendicular straight lines if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.

(d) Find the locus of a luminous point, if the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ casts a circular shadow on the plane $z=0$.

(e) Find the equations of the generators of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, through a point on the principal elliptic section Also find the angle between them.

(f) Reduce the equation $4x^2+4y^2+4z^2-2x-14y-22z+33=0$ to its canonical form and determine the type of the quadric represented by it.

4. Answer any one question 1X10

(a) (i) Find the condition for a system of forces to be in Astatic equilibrium. Also show that if a system of coplanar forces acting at different points of a body have a single resultant force and if each force be turned about its point of application through an angle θ, then their resultant of the new system will also turn about the Astatic centre through the same angle θ. (6)

(ii) A solid cone, of height h and semivertical angle 'α', is placed with its base against a smooth vertical wall and is supported by a string attached to its vertex and to a point in the wall; Show that the greatest possible length of the string is $h\sqrt{1+\frac{16}{9}\tan^2\alpha}$ (4)

(b)(i) A perfectly rough plane is inclined at an angle α to the horizon; Show that the least eccentricity of the ellipse which can rest on the plane is $\sqrt{\dfrac{2\operatorname{Sin}\alpha}{1+\operatorname{Sin}\alpha}}$ (6)

(ii) A solid hemisphere of weight 'W' rests in the limiting equilibrium with its curved surface on a rough inclined plane and its plane face is kept horizontal by a weight 'P' attached to a point in the rim Prove that the coefficient of friction is $\frac{P}{\sqrt{W(2P+W)}}$. (4)

5. Answer any one question 1×7

(a) A particle moves in a straight line with an acceleration towards a fixed point in the straight line which is equal to $\left(\frac{\mu}{x^2}-\frac{\lambda}{x^3}\right)$ at a distance x from the given point. It starts from rest at a distance a. Show that it oscillates between the distance 'a' and the distance $\frac{\lambda a}{2a\mu-\lambda}$ and the periodic time is $\frac{2\pi\mu a^3}{(2a\mu-\lambda)^{3/2}}$

(b) A heavy uniform flexible string of length $2l$ hangs over a small smooth pully The string is initially at rest with lengths $l+a$ and $l-a$ on the two sides of the pully. If the pully be now made to move upwards with a constant accleration 'f' then show that the string will leave the pully after a time $\sqrt{\frac{l}{f+g}}\ln\left(\frac{l+\sqrt{l^2-a^2}}{a}\right)$.

6. Answer any two questions 2×9

(a) (i) If two smooth spherical balls of masses m and m' moving with velocities u and u' respectively impinge directly, then prove that the condition that each losses the same amount of Kinetic energy is $(3+e)(mu+m'u')+(1-e)(mu'+m'u)=0$, where '$e$' is the coefficient of restitution. (6)

(ii) If the radial and transverse velocities of a particle be always proportial to each other, then show that the path is an equiangular spiral. (3)

(b) (i) Find the components of velocity and accleration of a moving point referred to a set of rectangular axes revolving with uniform angular velocity ω about the origin in their own plane. (5)

(ii) A particle is projected vertically upwards with a velocity 'u' and the resistance of the air produces a retardation kv^2, where v is the velocity. Show that the velocity u_1 with which the particle will return to the point of projection is given by $\frac{1}{u_1^2}=\frac{1}{u^2}+\frac{k}{g}$ (4)

(c) (i) If a particle is moving in a medium whose resistance varies at its velocity, then show that by a proper choice of axes, the equation of the trajectory can be put in the form $y+ax=b\log_e x$ (6)

(ii) An insect crawls at a constant rate u along the spoke of a cart wheel of radius a, the cart moving with a constant velocity v. Find the acceleration along and perpendicular to the spoke. (3)

(d) (i) A particle is projected under gravity in a medium whose resistance equals to mk times the velocity. Find the path of the particle if it be projected with a velocity 'u' at an angle 'α' to the horizon (5)

(ii) A particle moves with a constant accleration. Show that the space-average of the velocity over any distance is $\frac{2}{3}\frac{u_1^2+u_1u_2+u_2^2}{u_1+u_2}$ and the time average velocity is $\frac{1}{2}(u_1+u_2)$, where u_1 and u_2 are the initial and final velocities (4)