BANGABASI COLLEGE

B.Sc. Second Year (Part-II) Honours Test Examination-2015

Subject – Mathematics

Paper - I

Full Marks – 100

Time – 4 Hours

1. Answer any two questions: 2x4=8

(a) State Cauchy's condensation test of convergence of a series of positive real numbers. Use it to prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for p>1 and is divergent for p ≤ 1.

(b) Let I = [a, b] and a function $f : I \to$ IR be differentiable on I. Let f' (a) $\neq f'$ (b). If k be a real number lying between f' (a) and f' (b) then prove that there exists a point c in (a, b) such that f' (c) =k.

(c) Use Lagrange's mean value theorem to prove that $0 < \frac{1}{\log(1+x)} - \frac{1}{x} < 1$ for x > 0.

(d) State Leibnitz's theorem on successive differentiation on product of two functions. Using this theorem, prove that if $y = \tan^{-1} x$ then $2(n+1)xy_{n+1} + n(n+1)y_n = 0$, where $y_n = \frac{d^n y}{dx^n}$.

2. Answer any two questions: 2x5=10

(a) Solve the differential equation (any one) :

(i) $y(xy + 2x^2y^2)dx = x(xy - x^2y^2)dy = 0$
(ii) $(x^2y^3 + 2xy)dy = dx$.

(b) Reduce the equation $(px^2 + y^2)(px + y) = (p+1)^2$, $(p = \frac{dy}{dx})$ to Clairaut's form by using the substitution $u = xy$ *and* $v = x + y$ and hence find its complete primitive.

(c) Solve: $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2e^{3x}$

(d) Using the method of undetermined coefficients find the general solution of $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = (x-2)e^x$

3. Answer any one question: 1x7=7

(a) Verify that $y = e^x$ is a solution of the reduced equation of $(1-x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = (1-x)^2$. Solve the equation after reducing it to a linear equation of first order.

(b) Find, by the method of variation of parameters, the general solution of the equation $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 logx$, given that $y = x^3, y = \frac{1}{x}$ are two linearly independent solutions of the corresponding homogeneous equation.

4. Answer any three questions: 3x4=12

(a) A particle of mass m rests on a smooth horizontal plane and is attached to one end of a light elastic string, the other end of which is fastened to a fixed point on the plane. The unstretched length of the string being l, show that, if the particle be moved along the plane until its distance from the fixed point is l′ (> l), and is then let go, it will pass the fixed point after a time given by $t = \sqrt{\frac{ml}{\lambda}}(\frac{\pi}{2} + \frac{l}{l'-l})$

(b) A particle of mass m moving in a straight line is acted on by an attractive force $mk\frac{a^2}{x^2}$ for x ≥ a and $\frac{mkx}{a}$ for x< a, towards a fixed point on the line, x being the distance from the fixed point and k, a being constants. If the particle starts from rest at the point x = 2a, prove that it will reach the point x =0 with speed $\sqrt{2ka}$ after time $(1 + \frac{3\pi}{4})\sqrt{\frac{a}{k}}$.

(c) Prove that the mean kinetic energy of a particle of mass m moving under a constant force, in any interval of time is $\frac{1}{6}(u_1^2 + u_1u_2 + u_2^2)$, where u_1 *and* u_2 are initial and final velocities of the particle.

(d) Two spheres of masses M, m impinge directly when moving in opposite directions with speeds u, v respectively, and the sphere of mass m is brought to rest by collision. Prove that $v(m - eM) = M(1 + eu)$, where e is the coefficient of restitution.

(e) A thin straight smooth tube is made4 to revolve upward with a constant angular velocity ω in vertical plane about one extremity O. When it is in a horizontal position, a particle is at rest in it at a distance a from the fixed point O. If ω be small, then show that it will reach O in a time $(\frac{6a}{g\omega})^{\frac{1}{3}}$ approximately.

(f) A particle is thrown vertically upwards with a speed V. if the air resistance is assumed to vary as the square of the speed and to equal to gravity g when speed is U, then show that the particle will rise for a time $\frac{U}{g}\tan^{-1}\frac{V}{U}$

5. Answer any two questions: 4x2=8

[a] Express the following permutation $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 7 & 9 & 1 & 8 & 2 & 6 & 3 & 5 \end{pmatrix}$ as a product of disjoint cycles. Then express α as a product of transpositions. Is α an even transposition? 2+2+1

[b] Prove that the set of all even permutations of $\{1,2,3\}$ is a cyclic group. Determine two distinct generators of it. 2+2+1

[c] Prove that an infinite cyclic group has exactly two generators.

[d] Prove that the ring $(Z_p,+,\bullet)$ is an integral domain if and only if p is prime.

6. Answer any three questions: 5x3=15

[a] f(x,y) = $\begin{cases} x\sin\dfrac{1}{y}+\dfrac{x^2-y^2}{x^2+y^2} & if \quad y\neq 0 \\ 0 & if \quad y=0 \end{cases}$

Show that $\underset{y\to 0}{Lt}\ \underset{x\to 0}{Lt}\ f(x,y)$ exists but neither $\underset{(x,y)\to(0,0)}{Lt} f(x,y)$ nor $\underset{x\to 0}{Lt}\ \underset{y\to 0}{Lt}\ f(x,y)$ exists. 2+2+1

[b] Show that the function f, where f(x,y) = $\begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & if \quad x^2+y^2\neq 0 \\ 0 & if \quad x=y=0 \end{cases}$ 2+2+1

is continuous, possesses partial derivaties but is not differentiable at the origin.

[c] If u be a homogeneous function of degree n, (possessing continuous second order partial derivatives) then prove the following:- $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=n(n-1)u.$ Is the result valid if we assume the mere existence of second order partial derivatives ? Justify your answer.

4+1

[d] A function f(x,y) having continuous second order partial derivatives when expressed in terms of the new variables u and v defined by x = $\frac{1}{2}(u+v)$ and y^2 =uv becomes g(u,v); prove that

$$\frac{\partial^2 g}{\partial u\partial v}=\frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2}+2\frac{x}{y}\frac{\partial^2 f}{\partial x\partial y}+\frac{\partial^2 f}{\partial y^2}+\frac{1}{y}\frac{\partial f}{\partial y}\right).$$

1+4

[e] Show that the functions u = x+y-z, v = x-y+z, w = $x^2+y^2+z^2$-2yz are not independent. Find the relation between them.

[f] Write the conditions so that the functional equation f(x,y) = 0 does define an implicit function. Show that the equation xySinx + Cosy = 0 determine unique implicit function in the neighbourhood of the point $\left(0, \frac{\pi}{2}\right)$. Also find the first derivative of the solution.

2+2+1

7. Answer any two questions: 2× 4

(a) Show that the pedal equation of the parabola y^2 = 4a(x+a) is p^2 = ar .

(b)Show that for the cycloid x = a($\theta - \sin\theta$), y = a(1$-\cos\theta$) radius of curvature at any point is twice the length of the portion the normal intercepted between the curve and the axis of x.

(c)Show that the asymptotesof the curve

$(x^2-4y^2)(x^2-9y^2)+5x^2y-5xy^2-30y^3+xy+7y^2-1=0$ cut the curve in eight points which lie on a circle x^2+y^2 = 1.

(d) Prove that the curve y = $\log x$ is convex to the foot of the ordinate in the range 0< $x<1$ and concave when x> 1. Prove also that the curve is convex with respect to any point on the x axis.

8. Answer any one question: 2

(a) Find the area of the loop formed by the curve $a^2y^2 = x^3$(2a$-$x). πa^2

(b) Find the centroid of the arc of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ which lies in the first quadrant.

$\left(\frac{256a}{315\pi}, \frac{256a}{315\pi}\right)$

9. Answer any three questions: 3× 5

(a) A toy company manufactures two types of doll, a basic version –doll A and a akes a profit of adeluxe version-doll B.Each doll of type B takes twice as long to produce as one of type A and the company would have time to make a maximum 2000per day if it produces only the basic version. The supply of plastic is sufficient to produce 1500 dolls per day. The deluxe version requiresa fancy dress of which thereareonly 600 per day available. If the company makes a profit of Rs.3 and Rs.5 per doll respectively on doll A and B, then formulate the problem as a linear programming problem.

(b)Solve the following L.P.P. by graphical method

Minimize z = $20x_1 + 10x_2$

Subject to $x_1 + 2x_2 \leq 40$

$3x_1 + x_2 \geq 30$

$4x_1 + 3x_2 \geq 60$ $\quad x_1 = 6,\ x_2 = 12 \quad z_{min} = 240$

$x_1, x_2 \geq 0.$

(c) Find all the basic solutions of the following equations identifying in each case the basis vectors and the basic variables:

$$x_1 + x_2 + x_3 = 4$$

$$2x_1 + 5x_2 - 2x_3 = 3.$$

$\left(\frac{17}{3}, \frac{-5}{3}, 0\right)$ $\left(\frac{11}{4}, 0, \frac{5}{4}\right)$ $\left(0, \frac{11}{7}, \frac{17}{7}\right)$

(d) Prove that a basic feasible solution to a linear programming problem corresponds to an extreme point of the convex set of feasible solutions.

(e) $x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 0$ is a feasible solution of the system of equations

$$x_1 + 2x_2 + 4x_3 + x_4 = 7$$

$$2x_1 - x_2 + 3x_3 - 2x_4 = 4$$

Find a basic feasible solution. $\left(0, \frac{1}{2}, \frac{3}{2}, 0\right)$ or $(3, 2, 0, 0)$

(f) By solving the dual of the following problem , show that the given problem has no feasible solution

Minimize z = $x_1 - x_2$

Subject to $2x_1 + x_2 \geq 2$

$-x_1 - x_2 \geq 1$

$x_1, x_2 \geq 0$

10. Answer any one question: 3

(a) Find the optimal solution of the following transportation problem

	D_1	D_2	D_3	D_4	D_5	a_i
O_1	3	4	6	8	8	20
O_2	2	10	0	5	8	30
O_3	7	11	20	40	3	15
O_4	1	0	9	14	16	13
b_j	40	6	8	18	6	

$x_{11} = 20$, $x_{21} = 4$ $x_{23} = 8$, $x_{24} = 18$ $x_{31} = 9$, $x_{35} = 6$ $x_{41} = 7$
$x_{42} = 6$ min cost $= 246$

(b) Use dominance to solve the game problem with following payoff matrix

$$A \quad \overset{B}{\begin{array}{cccc} 3 & 2 & 4 & 0 \\ 2 & 4 & 2 & 4 \\ 4 & 2 & 4 & 0 \\ 0 & 4 & 0 & 8 \end{array}}$$

$(0, 0, \frac{2}{3}, \frac{1}{3})$ $(0, 0, \frac{2}{3}, \frac{1}{3})$ $V = \frac{8}{3}$

11. Answer any one question: 1x5=5

(a) A uniform heavy elliptic wire of semi-axes a, b is hung over a small rough peg. Show that if the wire can be in equilibrium with any point of it contact with the peg, the coefficient of friction must not be less than $\frac{a^2-b^2}{2ab}$.

(b) A solid hemisphere of weight W rests in limiting equilibrium with its curved surface on a rough inclined plane and the plane face is horizontal by a weight attached at a point in the rim. Prove that the coefficient of friction is $\frac{P}{\sqrt{W(2P+W)}}$.

12. Find the locus of the centre of the sphere which passes through the points $(0,0,\pm c)$ and cuts the lines $y = \pm x\tan\alpha, z = \pm c$ at two points A and B where AB has a constant length $2a$. 7

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