BANGABASI COLLEGE

B.Sc. Third Year Honours Test Examination-2016

Subject – Mathematics

Paper - I

Full Marks – 100

Time – 4 Hours

1. Answer any five questions · 5×5

(a) If S⊂ $\mathcal{R}$ be such that every infinite subset of S has a limit point in S then prove that S is a compact set. Hence show that $\mathcal{R}$ is not a compact set.

(b) If f : [a,b]→$\mathcal{R}$ has a derivative at every point x in [a,b] and f′ is bounded on [a,b] then prove that f is a function of bounded variation. Is the converse true? Justify your answer.

(c) Let f : [a,b]→$\mathcal{R}$ be a bounded function on [a,b]f. Prove that a necessary and sufficient condition of integrabilityof f on [a,b] is that for every ε>0 there exists a partition P of [a,b] such that U(P,f) – L(P,f)< ε.

(d) Prove that $\frac{\pi^3}{24\sqrt{2}} < \int_0^{\pi/2} \frac{x^2}{\sin x+\cos x} dx < \frac{\pi^3}{24}$.

(e) Let f : [a,b]→$\mathcal{R}$ be a function of bounded variation on [a,b] , prove that f is intregableon [a,b] .

(f) Let X be a compact subset of $\mathcal{R}$ and $f_n: X \to \mathcal{R}$ be a continuous function for each n∈ . If $\{f_n\}$ converges pointwise to a continuous function f on X and the sequence $\{f_n\}$ is monotone for every fixed x ∈ X and for each n ∈ $\mathcal{N}$ then prove that $\{f_n\}$ converges uniformly to f on X .

(g) If $\sum_1^\infty ne^{-nx}$ (x>0) converges to the function f for x> 0 , show that

(i) f is continuous for all x> 0 and (ii) $\int_{ln2}^{ln3} f(x)dx = \frac{1}{2}$.

(h) Given $(1+x)^{-1} = 1 - x + x^2 - \cdots$ ($|x| < 1$),deduce the power series expansion of $log_e(1+x)$ and establish its validity of expansion. Also deduce that

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \log 2$.

2. Answer any three questions: 6x3=18

[a] The matrix representation of a linear mapping T: $R^3 \to R^3$ be defined by $\begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

relative to the ordered basis (1,0,0),(0,1,0), (0,0,1). Find the explicit representation of linear mapping T and matrix representation of T relative to the basis { (1,1,1), (0,1,1), (0,0,1) }.

[b] Let V and W be finite dimensional vector spaces over a field F and T:V→W be a linear mapping. Prove that rank of T = rank of matrix of T.

[c] Prove that an infinite cyclic group is isomorphic to the additive group of integers.

[d] If N is a normal subgroup of a group G and G/N is the set of all left cosets of N in G,then prove that G/N is a group under the binary operation given by (aN)(bN) = (abN) $\forall a,b \in G$.
[Show that the operation is well defined and G/N is a group under this operation.]

[e] Define the christoffel symbols $\left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\}$. Prove that these are not components of any tensor.

[f] If A_{ij} is a covariant tensor and B^i a contravariant vector show that $A_{ij}B^j$ is a covariant vector.

3. Attempt any one from the following: 1x7=7

(a) A heavy uniform rod AB, of length 2a, rests with its ends in contact with two smooth inclined planes, of inclinations α and β to the horizon. If θ be the inclinations of the rod to the horizon, prove, by the principle of virtual work, that Tanθ =1/2(cot α - cot β).

(b) A cone of semi-vertical angle $\tan^{-1}(1/\sqrt{2})$ is enclosed in the circumscribing sphere; show that it will rest in any position.

4. Attempt all the questions: 2x4=8

(a) State Divergence theorem of Gauss. Verify Divergence theorem for F=xi + yj + zk over the the cylindrical region bounded by $x^2 + y^2 = 9$, z=0, z=2.

(b) Verify stokes theorem for $F = (2x-y)i - yz^2 j - y^2 z k$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

5 Answer any two questions: 2x4

(a) Use convolution theorem, find $L^{-1}\{\frac{1}{(p^2+4)^2 p}\}$.

(b) Solve the following differential equation using Laplace Transform
$(D^2 + 2D + 1)y = 3te^{-t}, t > 0$ given y=4, Dy= 2 at t= 0 where $D \equiv \frac{d}{dt}$.

(c) Solve the equation $\frac{d^2y}{dx^2} + (x-1)^2\frac{dy}{dx} - 4(x-1)y = 0$ in series near the ordinary point x=1.

6. Answer any two questions: 2x6=12

(a) State and prove principle of conservation of angular momentum for impulsive and finite forces.

(b A solid homogeneous cone of height h and vertical angle 2α oscillates about a horizontal axis through its vertex. Show that the length of the simple equivalent pendulum is $\frac{h}{5}(4 + tan^2\alpha)$.

(c) Three equal uniform rods AB, BC,CD are freely jointed and placed in a straight line on a smooth table. The rod AB is struck at its ends by a blow which is perpendicular to its length. Find the resulting motion and show that the velocity of the centre of AB is 19 times that of CD and its angular velocity 11 times that of CD.

(d) One end of a thread , which is wound in a reel is fixed and the reel falls in a vertical line , its axis being horizontal and the unwound part of the thread being vertical. If the reel b a solid cylinder of radius a and weight W, show that the acceleration of the centre of the reel is $\frac{2}{3}$g and the tension of the thread is $\frac{W}{3}$

7 Answer any two questions: 2x5 = 10

(a) A particle is moving with central acceleration $\mu(r^5 - c^4 r)$ being projected from from an apse at a distance c with a velocity $\sqrt{\frac{2\mu}{3}}c^3$ .Show that its path is the curve $x^4 + y^4 = c^4$

(b) A planet is describing an ellipse about the sun as focus; show that its ve4locity away from the sun is greatest when the radius vector to the planet is at right angles to the major axis of the path, and that it is $\frac{2\pi ae}{T\sqrt{1-e^2}}$, where 2a is the major axis, e the eccentricity of the orbit and T the periodic time of the planet.

(c) A particle oscillates in a smooth cycloid under gravity , the amplitude of motion being b, and period being T. Show that its velocity at time t measured from a position of a rest is $\frac{2\pi b}{T} sin\frac{2\pi t}{T}$.

(d) The volume of a spherical raindrop falling freely increases at each instant by an amount equal to μ times its surface area at that instant. If the initial radius of the drop be a , then show that its radius is doubled when it has fallen through a distance $\frac{9a^2g}{32\mu^2}$.

(e) Solve the following linear dynamical system:

$\dot{x} = x + y, \quad \dot{y} = 4x - 2y$ subject to the initial condition $(x_0,y_0) = (2,-3)$. Draw a phase portrait for the LDS and identify the critical point and discuss about its stability.

8 Answer any two questions: 2x6=12

(a) Find the differential equations of the curves of equipressure qand equidensity for a fluid in equilibrium under given external forces.

(b) When the depth of the fluid is increased by an amount a, the depth of the centre of pressure is found to be increased by y, and when instead, the depth of the liquid is increased by b, that of the centre of pressure is found to be increased by z. Show that the depth of the centre of gravity in the original state of liquid is $\frac{ab(b-a+y-z)}{(az-by)}$.

c) A vessel in the form of a paraboloid of revolution formed by the revolution of a parabola of latus rectum 4a about its axis, is filled to half its height with liquid. Show that the greatest angular velocity with which it can revolve about its axis so that no liquid is spilt is $\frac{1}{4}\sqrt{\frac{6g}{a}}$.

(d) Show that a homogeneous right circular cone of vertical angle 2α cannot float stably with its axis vertical and vertex downwards unless its density as compared with that of the liquid is greater than $\cos^6\alpha$. What is the corresponding result when the vertex is upwards?

(e) A gas satisfying Boyle,s law $p = k\rho$, is acted on by forces $X = \frac{-y}{x^2+y^2}, Y = \frac{x}{x^2+y^2}$ .Show that the density varies as $e^{\frac{\theta}{k}}$ where $tan\theta = \frac{y}{x}$

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BANGABASI COLLEGE

B.Sc. Third Year Honours Test Examination-2016

Subject – Mathematics

Paper - II

Full Marks – 75

Time – 4 Hours

1. Answer any two questions : 2× 5

(a) Show that $\int_0^{\frac{\pi}{2}} cos^p x sin^q x\, dx$ is convergent if and only if $p > -1, q > -1$.

(b) Show, with proper justification, that

$$\int_0^{\infty} e^{-x^2} cos ax\, dx = \frac{\sqrt{\pi}}{2} e^{-\frac{a^2}{4}} \text{ for all real } \alpha.$$

(c) Prove that $\iiint_E (lx + my + nz)^2 dxdydz = \frac{4\pi}{15}(l^2 + m^2 + n^2)$ where E is the region bounded by the sphere $x^2 + y^2 + z^2 = 1$.

2. Answer any three questions: 5x3=15

[a] Prove that a real function of a complex variable either has derivative zero or the derivative does not exist.

[b] Show that $f(z) = \sqrt{|xy|}$ is not analytic at the origin although Cauchy-Riemann equations are satisfied at that point.

[c] If $u(x,y) = x^3 - 3xy^2$, show that there exists a function $v(x,y)$ such that $f(z) = u(x,y) + iv(x,y)$ is analytic in a finite region of the complex plane.

[d] Let C [0, 1] denote the set of all real valued continuous functions on [0, 1]. For x,y $\in$C[0,1] define d(x,y) = $\int_0^1 |x(t) - y(t)|dt$. Show that d is a metric on C [0, 1].

[e] Given that (N,d) is a metric space where $d(m,n) = \frac{|m-n|}{mn}$ for all $m, n \in N$. Examine whether (N,d) is a complete metric space.

[f] Let P[0,1] be the set of all polynomials defined on [0,1]. Show that $d(p_1, p_2) = \sup_{0\le x\le 1}|p_1(x) - p_2(x)|$ is a metric on P[0,1]. Also show that this metric is incomplete.

3. Answer any five questions: 5x5=25

[a] Show that the remainder in approximating $f(x)$ by interpolation polynomial using distinct interpolating points $x_0, x_1, x_2, ... x_n$ is of the form $(x-x_o)(x-x_1)(x-x_2)........(x-x_n)\frac{f^{n+1}(\xi)}{(n+1)!}$ where $\min\{x, x_0, x_1, ... x_n\} < \xi < \max\{x, x_0, x_1, x_2, ... x_n\}$.

[b] Deduce numerical differentiation formula from Newton's Backward interpolation formula at an interpolating point.

[c] Deduce Simpson's 1/3rd Composite rule for numerical integration using Newton's forward interpolation formula and interpret it geometrically.

[d] Explain Newton-Raphson method for computing a simple root of an equation f(x) =0. Give its geometrical interpretation.

[e] Describe the power method to calculate numerically greatest eigen value of a real square matrix of order n.

[f] Write a FORTRAN program to find the sum of the series $y = x + \frac{x^3}{3} + \frac{x^5}{5} + ... + \frac{x^{15}}{15}$ where $x = 0.1, 0.3, 0.5, 0.7, 0.9$.

[g] Write a FORTRAN program to calculate $\int_{0.1}^{0.6} f(x)dx$ by Simpson's 1/3rd Rule when

$$f(x) = \frac{1+k}{\sqrt{1-\frac{1}{2}Cos^2 x}} \quad for \ \ k = 0.1(0.1)0.5.$$

[h] Given the length of three line segments a,b,c. Write an efficient FORTRAN program to test whether the line segments form a triangle. In case they form a triangle, test whether the triangle is obtuse angled, right angled or acute angled.

4. Answer any 3 from the following questions: 3X3=9

(a) Show that the second order mean about any point is minimum when taken about the mean.

(b) If A and B are two independent random events then show that $\bar{A}$ and $\bar{B}$ are also so.

(c) There are two urns containing white and black balls. The first urn contains 2 white and 3 black balls, the second urn contains 3 white and 5 black balls. An urn is chosen at random and a ball is drawn from it. Find the probability that the ball drawn is white.

(d) Show that a function given by $f(x) = x, \quad 0 < x < 1$

$$= k - x, \quad 1 < x < 2$$
$$= 0, \quad elsewhere.$$

is a p.d.f for a suitable value of k. Calculate the probability that the random variable lies between $\frac{1}{2}$ and $\frac{3}{2}$.

(e) State De-Moivre-Laplace limit theorem. Write the Characteristic function for Poisson distribution.

5. Answer any 2 from the following questions:- **4X2=8**

(a) Find the limit of co-rreletion co-efficient.

(b) If $P(X = a) > 0$ (where a is a real number), then show that $F(x)$ has a jump discontinuity on the left at the point $x = a$, the height of jump being equal to $P(X = a)$.

(c)) A point is chosen at random on a semi-circle having centre at the origin and radius unity and projected on the diameter. Prove that the distance of the point of projection from the centre has probability dnsity

$$f(x) = \frac{1}{\pi\sqrt{1 - x^2}} \; for - 1 < x < 1$$
$$= 0, elsewhere$$

6. Answer any 1 from the following questions:- **8X1=8**

(a) Prove that the statistic $t = \frac{\sqrt{n}(\bar{X} - m)}{s}$, known as student's ratio is t-distributed with $\nu = n - 1$ degree of freedom.

(b) In a random sample of 400 articles 40 are found to be defective. Obtain 95% confidence interval for the true proportion of defectives in the population of such articles.

Given : $\int_0^{1.96} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 0.475$.

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